Ex. 2.7.2

(a) $0 \le \frac{1}{2^n+n} \le \frac{1}{2^n}$ & $\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-1/2} = 2$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n} = 2 - 1 = 1$

is convergent

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n+n}$ is convergent

by the Comparison Test.

Ex. 2.7.2

(d) $$\sum_{k=0}^{\infty}\left(\frac{1}{3k+1}+\frac{1}{3k+2}-\frac{1}{3k+3}\right)$$

$$=\sum_{k=0}^{\infty}\left(\frac{(3k+1)+(3k+2)}{(3k+1)(3k+2)}-\frac{1}{3k+3}\right)$$

$$=\sum_{k=0}^{\infty}\frac{\big((3k+1)+(3k+2)\big)(3k+3)-(3k+1)(3k+2)}{(3k+1)(3k+2)(3k+3)}$$

factor out

$$=\sum_{k=0}^{\infty}\frac{(3k+1)(3k+3)+\overbrace{(3k+2)}(3k+3)-(3k+1)\overbrace{(3k+2)}}{(3k+1)(3k+2)(3k+3)}$$

$$=\sum_{k=0}^{\infty}\frac{(3k+1)(3k+3)+(3k+2)\overbrace{\big((3k+3)-(3k+1)\big)}^{=2}}{(3k+1)(3k+2)(3k+3)}$$

$$=\sum_{k=0}^{\infty}\frac{\cancel{(3k+1)}\cancel{(3k+3)}}{\cancel{(3k+1)}(3k+2)\cancel{(3k+3)}}+\frac{2\cancel{(3k+2)}}{(3k+1)\cancel{(3k+2)}(3k+3)}$$

$$=\sum_{k=0}^{\infty}\frac{1}{3k+2}+\frac{2}{(3k+1)(3k+3)}$$

$$\geq\sum_{k=0}^{\infty}\frac{1}{3k+2}$$

$$\geq\sum_{k=0}^{\infty}\frac{1}{3k+3}=\frac{1}{3}\sum_{k=1}^{\infty}\frac{1}{k}=\infty$$

Ex. 2.7.2

(e) $1-\frac{1}{2^2}+\frac{1}{3}-\frac{1}{4^2}+\frac{1}{5}-\frac{1}{6^2}+\cdots$

$$\underbrace{1-\frac{1}{2^2}}_{=\frac{2^2-1}{2^2}=\frac{3}{4}} \qquad \underbrace{\frac{1}{3}-\frac{1}{4^2}}_{=\frac{4^2-3}{4^2\cdot 3}=\frac{13}{16\cdot 3}=\frac{13}{48}} \qquad \underbrace{\frac{1}{5}-\frac{1}{6^2}}_{=\frac{6^2-5}{6^2\cdot 5}=\frac{31}{36\cdot 5}=\frac{31}{180}}$$

and in general

$$\frac{1}{2k+1}-\frac{1}{(2(k+1))^2}=\frac{4(k+1)^2-(2k+1)}{4(k+1)^2(2k+1)}$$

$$=\frac{4k^2+8k+4-2k-1}{8k^3+17k^2+16k+4}$$

$$\geq \frac{4k^2-2k-1}{(8+17+16+4)k^3}$$

$$=\frac{4k^2-2k-1}{45k^3}$$

$$\geq \frac{\cancel{4}^{\,2}k^2}{45k^3}$$

$$=\frac{\cancel{4}^{\,2}}{45}\cdot\frac{1}{k}$$

so diverges by Comparison Test.

$4k^2-2k-1 \geq 2k^2$ if $k\geq 2$

since $2x^2-2x-1$ has roots

$x=\frac{2\pm\sqrt{4+8}}{4}$ are

$\leq \frac{2+6}{4}$

$=2$

~~roots of~~ $\cancel{4x^2-2x-1}$

$\cancel{x=\frac{2\pm\sqrt{4+16}}{8}}$

$\cancel{=\frac{2\pm2\sqrt{5}}{8}}$

$\cancel{\leq \frac{2+2\sqrt{5}}{8}\approx 0.81}$

Ex. 2.7.9 (Ratio Test)

Given $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$ and with

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$$

If $r < 1$, the series converges absolutely, while if $r > 1$, the series diverges. If $r = 1$, no information is given.

(a) pf: Since $\frac{a_{n+1}}{a_n} \to r$, if $r < 1$, then

$$\forall s \in (r, 1),\ \exists N \in \mathbb{N} \text{ s.t. } n \geq N$$

$$\Longrightarrow \frac{a_{n+1}}{a_n} < s$$

because if we choose $\varepsilon = s - r$, $\exists N \in \mathbb{N}$ s.t.

$$n \geq N \Longrightarrow \left|\frac{a_{n+1}}{a_n} - r\right| < \varepsilon \Longrightarrow \frac{|a_{n+1}|}{|a_n|} < \varepsilon + r$$

$$= s - r + r$$

$$= s$$

by reverse triangle ineq.

Therefore,

(a)

$$n \geq N \implies |a_{n+1}| < s|a_n| < s^2|a_{n-1}| \;\vdots\; < s^{n-N+1}|a_N| = s^n \cdot \frac{|a_N|}{s^{N-1}}$$

(b) But

$$0 < s < r < 1 \implies \frac{|a_N|}{s^{N-1}} \sum_{n=1}^{\infty} s^n = \frac{|a_N|}{s^{N-1}} \cdot \frac{s}{1-s}$$

converges

(c) and

$$\sum_{n=1}^{m} |a_n| \leq \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n|$$

$$\leq \sum_{n=1}^{N-1} |a_n| + \frac{|a_N|}{s^{N-1}} \cdot \frac{s}{1-s}$$

So the partial sums $s_m := \sum_{n=1}^{m} |a_n|$ are bounded and increasing. The MCT says $s_n \to s$ converges, i.e. $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.